Embedding of Quantified Higher-Order Nominal Modal Logic into Classical Higher-Order Logic*

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Abstract

In this paper, we present an embedding of higher-order nominal modal logic into classical higher-order logic and study its automation. There exists no automated theorem prover for first-order or higher-order nominal logic at the moment, hence, this is the first automation for this kind of logic. In our work, we focus on nominal tense logic and have successfully proven some first theorems.

1 Introduction

In writing ATP systems for non-classical logics, it is a common approach to develop special calculi and prover for each logic. Apart from the implementation work, this imposes the need to prove completeness and soundness for each calculus individually. A different approach is to embed a logic in higher-order logic (HOL), which had recently been a notable success (see, e.g., [1]). A major benefit is that there is no need to build new ATPs, but only to write a translation from one calculus to another.

In this paper we modified an embedding based approach for ordinary modal logic (see [2, 1]) and employed it for the embedding of higher-order nominal modal logic (HONL). Surprisingly, in our translation we observed a problem with the valuation of nominals. We propose two solutions to this problem. Lastly, we implemented the embedding for a special kind of HONL, the higher-order nominal tense logic (HONTL), a higher-order version of the nominal tense logic used by Blackburn [9].

As a novelty, our approach (up to our knowledge) implements the first reasoner for higher-order nominal logic. Already existing ATPs, such as hylotab [4], htab [7] and spartacus [5] are restricted to propositional nominal logic.

In section 2 we briefly present the HONL syntax and semantics including HONTL as a special case. In section 3, we survey the embedding of ordinary modal logic into HOL to encourage our own embedding. Finally, we present the embedding of HONL and some of our tests using the Isabelle/HOL proof assistant [8].

2 Higher-Order Nominal Logic

Nominal (modal) logic, often referred to as *hybrid logic*, is a general term for extensions of ordinary modal logics. The nominal logic considered here, introduces a new sort of constant symbols, the so-called *nominals*, that are only true at one possible world and false at every other. This logic is often denoted $\mathcal{H}$ and is the simplest of the nominal extensions [3].

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Early forms of nominal logic originated from Arthur Prior's research on tense logics [10] and were further developed by Robert Bull [3]. Since then, hybrid logics have intensively been studied by several others, including Valentin Goranko [11] and Patrick Blackburn [3, 9].

In contrast to most hybrid logic literature, we consider a simple extension to higher-order modal logics (HOML), rather than to propositional variants. The resulting logic is denoted HONL (for *higher-order nominal logic*). We closely follow the notation for higher-order logics used in [1], where also a brief introduction can be found.

Definition 1 Let I be some index set and T the set of simple types, freely generated from $\{o, \mu\}$ (where o is the type of Booleans and μ the type of individuals) and the right-associative function type constructor $\rightarrow$. The grammar for HONL is given by ($\alpha, \beta \in T, i \in I$):

$$\begin{aligned} s,t ::= \;\; & p_\alpha \mid n_o \mid X_\alpha \mid (\lambda X_\alpha . s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta} t_\alpha)_\beta \mid (\neg_{o\rightarrow o} s_o)_o \mid \\ & ((\vee_{o\rightarrow o\rightarrow o} s_o) t_o)_o \mid \forall_{(\alpha\rightarrow o)\rightarrow o}(\lambda X_\alpha . s_o))_o \mid (\Box^i_{o\rightarrow o} s_o)_o \end{aligned}$$

The symbols p_α, X_α and n_o denote constant symbols, variables, and nominals respectively. Further operators such as $\wedge, \rightarrow, \leftrightarrow, ...$ can be defined in the usual way. The terms of type o are called *formulas*.

The semantics can be adopted in large parts from ordinary higher-order modal logics, except that we restrict the interpretation of nominals:

A *model* M for HONL is a pair $M = \langle W, \{R_i\}_{i\in I}, \mathcal{D}, \{\mathcal{I}_w\}_{w\in W}\rangle$ where W is the (non-empty) set of possible worlds with each R_i being a accessibility relation between them. $\mathcal{D}$ is a collection of sets $\mathcal{D}_\alpha$, for each $\alpha \in T$, with $\mathcal{D}_o = \{T, F\}$ (for truth and falsehood) and $\mathcal{D}_{\alpha\rightarrow\beta}$ a set of functions from $\mathcal{D}_\alpha$ to $\mathcal{D}_\beta$. Each $\mathcal{I}_w$ is an interpretation mapping each p_α to its denotation in $\mathcal{D}_\alpha$ (depending on the world w it is interpreted in). As a special case, every nominal n_o is assigned a value in $\{T, F\}$, such that each n_o is mapped to T by exactly one $\mathcal{I}_w$ and to F by every other interpretation.

A valuation g assigns each variable X_α to an object in $\mathcal{D}_\alpha$. An X-variant of g is a valuation $g[a \,/\, X_\alpha]$, that maps each symbol $Y_\alpha \neq X_\alpha$ to $g(Y_\alpha) \in \mathcal{D}_\alpha$, except that X_α is mapped to $a \in \mathcal{D}_\alpha$. We assume the usual β- and η-reduction with the associated β- and $\beta\eta$-normal form.

Definition 2 The value of s_o in world w and model M, with valuation g, is denoted $\|s_o\|^{M,g,w}$ and defined by:

1. $\|p_\alpha\|^{M,g,w} = \mathcal{I}_w(p_\alpha)$ and $\|n_o\|^{M,g,w} = \mathcal{I}_w(n_o)$
2. $\|X_\alpha\|^{M,g,w} = g(X_\alpha)$
3. $\|(s_{\alpha\rightarrow\beta} t_\alpha)\|^{M,g,w} = \|s_{\alpha_\beta}\|^{M,g,w}(\|t_\alpha\|^{M,g,w})$
4. $\|\lambda X_\alpha . s_\beta\|^{M,g,w} = f \in D_{\alpha\rightarrow\beta}$ s.t. $\forall d \in D_\alpha : \; f(d) = \|s_\beta\|^{M,g[d \,/\, X_\alpha],w}$
5. $\|(\neg s_o)\|^{M,g,w} = T$ iff $\|s_o\|^{M,g,w} = F$
6. $\|\forall_{(\alpha\rightarrow o)\rightarrow o}(\lambda X_\alpha . s_o)\|^{M,g,w} = T \Leftrightarrow \forall d \in D_\alpha : \; \|s_o\|^{M,g[d \,/\, X_\alpha],w} = T$
7. $\|\Box^i_{o\rightarrow o} s_o\|^{M,g,w} = T \Leftrightarrow \forall v \in W : \; R_i \, w \, v \Rightarrow \|s_o\|^{M,g,v}$

We use the same definition for validity (in models) as given in [1].

The constructed language HONL corresponds to the nominal extension of modal logic K. As usual, we can enrich the conditions on frames by introducing properties on the accessibility relations (e.g. transitivity or symmetry).

The extension with nominals allows us to express formulas that explicitly talk about the world they are evaluated in. As an example, consider the formula "If I am at world w, P_o must hold". This formula cannot be expressed in modal logic – but if we have a nominal w_o (that is appropriately mapped), we can express it simply as $w_o \rightarrow P_o$. Furthermore, with the help of nominals, we can express certain frame conditions that cannot be formulated in ordinary modal logic, e.g. irreflexity or antisymmetry [9]. Thus, without any further extensions, HONL is already more expressive than HOML.

Often, a logic with special modal operator L (called *shifter*) with S5 context is considered [9]. With this operator, formulas like $L(1941_o \rightarrow \text{(Zuse completed Z3)}_o)$ can easily be formulated. Hence, L shifts to the world denoted by 1941_o and checks whether the given formula is true in exactly this world.

Nominal Tense Logic Higher-Order Nominal Tense Logic (HONTL), the higher-order variant of *Tense logic* K_t [6], can be interpreted as a restricted version of HONL. Here, we introduce two modal operators G and H (for $G = \Box_1, G = \Box_2$), as *goes on* and *has been*, with the dual operators $F = \neg G \neg$ and $P = \neg H \neg$ as *in the future* and *in the past* respectively.

Together with the two axioms $p \rightarrow HFp$ and $p \rightarrow GPp$, HONTL is already completely characterized.

3 Embedding

Ordinary Modal Logic. The embedding of HOML in HOL was described by Benzmüller and Woltzenlogel Paleo [1]. The main idea is to lift the truth (of type o) of a formula to be dependent on the world it is evaluated in (to a type $\sigma := \iota \rightarrow o$) and to introduce possible worlds as dedicated objects of a new type ι.

Definition 3. Let $\lceil . \rceil$ be the function that type-raises HOML terms. For each accessibility relation i a new constant symbol $r^i_{\iota \rightarrow \iota \rightarrow o}$ is introduced. The lifting translations of the connectives are as follows

$$\begin{aligned}
\lceil \neg \rceil_{\sigma \rightarrow \sigma} &= \lambda s_\sigma . \lambda W_\iota . \neg (s\, W) \\
\lceil \vee \rceil_{\sigma \rightarrow \sigma \rightarrow \sigma} &= \lambda s_\sigma . \lambda t_\sigma . \lambda W_\iota . (s\, W) \vee (t\, W) \\
\lceil \forall \rceil_{(\alpha \rightarrow \sigma) \rightarrow \sigma} &= \lambda s_{\alpha \rightarrow \sigma} . \lambda W_\iota . \forall (\lambda X_\alpha . s\, X\, W) \\
\lceil \Box^i \rceil_{\sigma \rightarrow \sigma} &= \lambda s_\sigma . \lambda W_\iota . \forall V_\iota . \neg (r^i_{\iota \rightarrow \iota \rightarrow o}\, W\, V) \vee s\, V
\end{aligned}$$

As it can be seen above, the embedding makes the possible worlds explicit in the HOL definitions. The predicate **valid** $= \lambda s_\sigma . \forall (\lambda W_\iota . s\, W)$ realizes the validity of a formula s_σ in HOL. As abbreviation, the notion $[s_\sigma]$ is used. The embedding of HOML is sound and complete, i.e. $\models_{HOML} s_o \Leftrightarrow \models_{HOL} [\lceil s_o \rceil]$, as shown in [1].

HONL Embedding. In order to expand the embedding of HOML to HONL, we only have to address the valuation of nominals. The only change in the semantics is that the valuation of nominals maps to singleton sets of worlds, i.e. is a function from nominals to worlds.

Where in the HONL syntax nominals are objects of type o, their lifted equivalents are assigned a new distinct type η. We hereby emphasize that the truth-value of a nominal n_η depends on the world it is evaluated in. The lifting of these objects is given by $\lceil n_o \rceil = \langle n_\eta \rangle = \lambda W_\iota . \lambda V_{\eta \rightarrow \iota} . (W = V n_\eta)$. For the valuation used inside the lifting definition we studied two approaches:

1. We choose the valuation $V_{\eta\to\iota}$ to be a globally fixed function, called $\mathit{worldAt}_{\eta\to\iota}$. We use the above approach, but replace the lifting immediately to $\langle n_\eta \rangle = \lambda W_\iota.(W = \mathit{worldAt}\, n_\eta)$.
2. As for worlds, we pass a function $V_{\eta\to\iota}$ through the formula. Each term is further lifted to take not only worlds, but nominal valuation functions as well. The validity predicate is then adjusted to be
$$\textbf{valid} = \lambda s_{\iota\to(\eta\to\iota)\to o}.\forall W_\iota.\forall V_{\eta\to\iota}.s\, W\, V$$

Both approaches abstract the valuation of nominals out of the formula, by introducing the valuation as part of the embedding, hence, reducing it to HOML.

The second approach clearly preserves the semantics, for the quantification over all valuations is done explicitly. The first approach does only work if we (a) work with a fixed valuation or (b) quantify over all nominals. For (b), to preserve the semantics, we have to require surjectivity for *worldAt*. The main idea behind this interpretation is that a quantification over all valuations will ultimately lead a quantified nominal to be true at each world at least once. Hence, we do not lose generality.

4 Tests

We used Isabelle/HOL [8] to implement the embedding described in Section 3. As for the theoretical part, we modified the Isabelle/HOL embedding of Benzmüller and Woltzenlogel Paleo[1] for nominal logic. In this section, we present some tests of our embedding[2].
The first test was to check some frame-condition correspondences (see [9]) that can now be expressed in our approach. Figure 1 shows the formulation of the respective theorems for irreflexivity and antisymmetry correspondences (the operators with a `t`-prefix, e.g. `t⟶`, are the representation of the associated type-lifted operators and `t<` is the accessibility relation).

```
lemma L1a:
  assumes "∀i.[<i> t→ t¬ (F <i>)]"
  shows "∀x.¬(x t< x)"

lemma L1b:
  assumes "∀x.¬(x t< x)"
  shows "∀i.[<i> t→ t¬ (F <i>)]"

lemma L2a:
  assumes "∀i.[<i> t→ t¬ (F (F <i>))]"
  shows "∀x.∀y. x t< y ⟶ ¬(y t< x)"

lemma L2b:
  assumes "∀x.∀y. x t< y ⟶ ¬(y t< x)"
  shows "∀i.[<i> t→ t¬ (F (F <i>))]"
```

Figure 1: Isabelle formulation of irreflexity (left), antisymmetry (right)

These correspondences were proven in both possible interpretations for the valuation of nominals, except for one lemma. Interestingly, this one was able to be proven by metis itself (with hints) but several other ATPs via sledgehammer gave up. Surprisingly the direction `L*b` is faster in the quantified approach, but without hints it is impossible to proof the `L*a` direction. The time for proving these lemmas was mostly in the area of 10ms with the fixed valuation (variant (1)) and took up to 840ms (right direct correspondence). The second valuation method took 3-4ms in `L*b` direction but around 40ms for the `L*a` direction with hints.

Next, we tested the framework on a simple example with quantification. We introduced an individual W(ashington) and stated that this person was not president at the Independence

[1]The original Isabelle embedding by Benzmüller and Woltzenlogel Paleo can be found at `https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Isabelle`

[2]Our embedding can be found at
`http://mwisnie.userpage.fu-berlin.de/logic/honl-embedding/nominal-embedding.tar`

```
consts indDayTime :: "ι"
   federalHallTime :: "ι"
   independenceDay :: "η"
   federalHall :: "η"
   Pr :: "μ ⇒ σ"
   W :: "μ ⇒ σ"

consts w :: "μ"
axiomatization where
   washington: "[W w]" and
   w1: "¬((Pr w)indDayTime)" and
   w2: "(Pr w)federalHallTime"
```

```
lemma WwillAlwaysHaveBeenPresident:
   shows "[L(<federalHall> t→ Pr w)]"

lemma WwasPresident:
   shows "[∃(λx.L(<federalHall> t→(Pr x)))]"

lemma WwasnotPresidentBefore:
   shows "[∃(λx.L(<federalHall>t→P(t¬(Pr x))))]"
```

Figure 2: Isabelle example problem: When was Washington president?

Day, but after his inauguration he was, of course, president. The problems in Figure 2 were successfully proven automatically. In the proof context we keep the valuation fixed by the defined axioms. Therefore, we chose the first approach and formulated the valuation as a globally fixed function.

5 Conclusion

We were able to adopt the embedding of HOML to HONL and thus use existing higher-order ATP to reason about nominal logic. The first tests were promising: Basic correspondences and tests were automatically proven. To give a full evaluation of this attempt, more experiments have to be carried out. Especially the pros and cons of both presented approaches and possible other options have to be evaluated.

References

[1] C. Benzmüller and B. Woltzenlogel Paleo. Automating Gödel's ontological proof of god's existence with higher-order automated theorem prover. In *21st European Conference on Artificial Intelligence (ECAI)*, 2014. (available at http://page.mi.fu-berlin.de/cbenzmueller/papers/C40.pdf).

[2] C. Benzmüller and L. Paulson. Quantified multimodal logics in simple type theory. *Logica Universalis (Special Issue on Multimodal Logics)*, 7(1):7–20, 2013.

[3] P. Blackburn, J.F.A.K. van Benthem, and F. Wolter. *Handbook of Modal Logic*. Studies in Logic and Practical Reasoning. Elsevier Science, 2006.

[4] J. V. Eijck. Hylotabtableau-based theorem proving for hybrid logics. Technical report, 2002.

[5] D. Götzmann et al. Spartacus: A tableau prover for hybrid logic. *Electronic Notes in Theoretical Computer Science*, 262(0):127 – 139, 2010.

[6] A. Galton. Temporal logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Fall 2008 edition, 2008.

[7] G. Hoffmann and C. Areces. Htab: a terminating tableaux system for hybrid logic. *Electr. Notes Theor. Comput. Sci.*, 231:3–19, 2009.

[8] T. Nipkow, L. C. Paulson, and M. Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer, 2002.

[9] P.Blackburn. Nominal tense logic. *Notre Dame Journal of Formal Logic*, 34(1):56–83, 1993.

[10] A.N. Prior. *Papers on time and tense*. Oxford books. Clarendon P., 1968.

[11] V.Goranko. Temporal logic with reference pointers. In *Proceedings of the 1st International Conference on Temporal Logic, volume 827 of LNAI*, pages 133–148. Springer.